Eur. Phys. J. B **38**, 223–230 (2004)
DOI: 10.1140/epjb/e2004-00115-0

THE EUROPEAN PHYSICAL JOURNAL B

Number of cycles in off-equilibrium scale-free networks and in the Internet at the Autonomous System Level

G. Bianconi[a]

The Abdus Salam International Center for Theoretical Physics, Strada Costiera 14, 34014 Trieste, Italy

Received 15 October 2003 / Received in final form 15 January 2004
Published online 14 May 2004 –

Abstract. In order to characterize networks in the scale-free network class we study the frequency of cycles of length h that indicate the ordering of network structure and the multiplicity of paths connecting two nodes. In particular we focus on the scaling of the number of cycles with the system size in off-equilibrium scale-free networks. We observe that each off-equilibrium network model is characterized by a particular scaling in general not equal to the scaling found in equilibrium scale-free networks. We claim that this anomalous scaling can occur in real systems and we report the case of the Internet at the Autonomous System Level.

PACS. 89.75.-k Complex systems – 89.75.Hc Networks and genealogical trees

1 Introduction

Many systems in nature and society can be described in terms of networks in which the elements of the system are represented as nodes and their interaction as links. It has been found that there is a large class of these networks that share common properties of the connectivity distribution [1–3]. In fact very different networks have been found to be scale-free i.e. with a power-law connectivity distribution $P(k) \sim k^{-\gamma}$. These networks are as different as the Internet, the scientific citation network or the network of the protein-protein interactions in the cell (which has a cutoff at large value of k in the connectivity distribution). Modeling the dynamic under which real networks evolve has been a recent challenge of the statistical mechanics community. Networks models can be classified into equilibrium and off-equilibrium models [3]. Equilibrium networks are characterized by having a constant number of nodes (and constant average connectivity) while off-equilibrium networks evolve under the continuous addition of new nodes and links. While certainly the connectivity distribution is responsible for the common properties of scale-free networks, as the robustness under random damage for example [4], other quantitative measurements are needed to characterize and distinguish between them. One direction is certainly to measure the correlations between the degree of connected nodes. Consequently the average connectivity $k_{nn}(k)$ [5,6] of the nearest neighbor of a node of degree k and the correlations plots [7,8] have been introduced. Another direction tries to characterize the ordering of the network by counting the frequency of cliques (triples of fully connected nodes) or of grid-like structures (evaluating the frequency of rectangular cycles in the network). At this scope it has been introduced the clustering coefficient $c_{3,i}$ [9] and more recently the grid coefficient $c_{4,i}$ [10]. The dependence of these coefficients on the connectivity k_i of node i give a measure of the modularity of the network [5,6,11]. Finally some attention has been addressed to the characterization of the networks in terms of the subgraphs recurring more frequently. In particular the ratio between the recurrence of a given subgraph on a particular real world network and the recurrence of the same subgraph in a network with the same degree distribution but randomly rewired has been computed for different real networks [12]. The subgraphs for which this ratio is higher than one are called characteristic motifs of the studied network. In the same spirit of [13] in this paper we will focus on a particular class of subgraphs: the cycles. We define a cycle of size h (a h-cycle) as a closed path of h links that visits each intermediate node only once. Such subgraphs are of special interest because they give an indication on the ordering of the networks and on the multiplicity of paths connecting two nodes of the network. In particular we will study the anomalous scaling of the number of cycles with the system size in off-equilibrium scale-free networks as compared with the results [14] for equilibrium scale-free networks. The paper is organized as follows: in Section 2 we give an heuristic calculus of the scaling of the number of cycles in the BA network [15,16] (which is the prototype of off-equilibrium networks models), in Section 3 the scaling of the number of cycles with the system size is studied numerically in different off-equilibrium scale-free models, in Section 4 we consider the real evolution of the Internet at the

[a] e-mail: gbianconi@ictp.trieste.it

Autonomous System Level, a classical example of scale-free network [5,6,8,17–19]; finally the conclusions are given in Section 5.

2 Analytic results

Much work [20] has been done regarding the number of subgraphs in Erdös and Renyi (ER) random networks [21] with Poisson degree distribution. On the contrary a recent work [14] present an analytic estimation of the number of given subgraphs in scale-free networks. We recall here the result relevant to our study omitting the proof that can be found in the cited paper. In maximally random (equilibrium) scale-free networks with power-law connectivity distribution $P(k) \sim k^{-\gamma}$ and fixed average degree the authors of [14] demonstrate that the number N_h of cycles of length h scales like a power-law of the system size

$$N_h(N) \sim N^{\xi} \tag{1}$$

with

$$\xi = \begin{cases} 1 & \text{for} \quad \gamma \le 2 \\ 3-\gamma & \text{for} \quad 2 < \gamma \le 3 \\ 0 & \text{for} \quad \gamma \ge 3. \end{cases} \tag{2}$$

The maximal exponent $\xi = 1$ is reached when $\gamma \le 2$ in which the network is "condensed" and the topology of the network is dominated by the hub,i.e. the node with maximal number of connections. We observe here that the exponent ξ of the scaling (1) of the number of cycles N_h does not depend on the length h of the cycle and is bounded by the unitary value, i.e.

$$\xi \le 1. \tag{3}$$

(On the contrary a simple combinatorial calculation prouves that the maximal number of possible triangles scales like $\langle k \rangle^h$ if no restrictions to the connectivity distribution are assumed.) What would be the scaling of N_h in off-equilibrium scale-free networks? We first try to answer this question addressing the problem for the prototype of off-equilibrium networks, the BA [15,16] network.

2.1 Number of cycles in the BA network

The BA model [15,16] was the first and simplest algorithm generating scale-free networks by an off-equilibrium dynamics. In this model, a new node is added to the network at each time step, and it is connected by a fixed number of links m to highly connected existing nodes (preferential attachment). According to this rule, the probability that at time t a new link will connect the new node with an existing node i of the network is assumed to be proportional to the degree $k_i(t)$ of node i. Multiple links are not allowed. If a double link occurs it is regarded as a single link and an additional link is extracted departing from the new node.

The model can be easily analyzed by a mean field approximation [15]. By this approach one finds that the average degree of a node i that entered the network at time t_i increases with time as a power-law

$$k_i(t) = m\sqrt{\frac{t}{t_i}}. \tag{4}$$

A network built in this way displays a power law degree distribution $P(k) \sim k^{-\gamma}$ with $\gamma = 3$.

In the case $m > 2$, the BA scale-free network is a very compact network, with cycles of any size. As the network evolves, new cycles are introduced in the network. By definition, new cycles include the newly added node: indeed, a new h-cycle is formed if the new node is connected to two nodes already connected by a self-avoiding path of size $h-2$. We indicate with $p_{i,k}$ the probability that the nodes i, k, attached to the network at time t_i, t_k, are connected by a link. The rate at which new cycles of length h are formed is given by the probability $p_{i,k}p_{j,k}$ that the new node k is linked to two existing nodes i and j times the probability $P_{i,j}^{h-2}(t)$ that they are already connected by a self-avoiding path of size $h-2$.

Therefore, we write the following rate equation for the average number of h-cycles $N_h(t)$

$$\frac{\partial \langle N_h(t) \rangle}{\partial t} = \frac{1}{2}\sum_{i=1}^{t}\sum_{j=1}^{t} p_{i,k}p_{j,k}P_{ij}^{h-2}(t), \tag{5}$$

where the factor $\frac{1}{2}$ takes into account that each pair of nodes i, j has been counted twice in the sums.

On the other hand, two nodes belong to a h-cycle if there is a link between them and, besides it, they are connected by a self-avoiding path of length $h-1$.

Let $P_{i,j}^{h-1}(t)$ be the probability that this path exists; thus, the probability that the link between node i and j belongs to a h-cycle is given by $p_{i,j}P_{i,j}^{h-1}(t)$. We obtain the average number $\langle N_h(t) \rangle$ of h-cycles in the system times $2h$ by summing this quantity over all the nodes i, j in the network. In fact, each cycle has been counted $2h$ times, because there are h nodes in the cycle, and two possible directions. Therefore, we can write

$$\langle N_h(t) \rangle = \frac{1}{2h}\sum_{i=1}^{t}\sum_{j=1}^{t} p_{i,j}P_{ij}^{h-1}(t). \tag{6}$$

Neglecting the fact that multiple links are not allowed we can assume in first approximation that at each time the extraction of the m nodes to which the new node will connect are independent random processes. In this approximation the probability $p_{i,k}$ that the node k arrived in the network at time t_k will be connected to a node i is given by

$$p_{i,k} = m\frac{k_i(t_k)}{\sum_j k_j(t_k)}. \tag{7}$$

By replacing equation (4), valid asymptotically in time, in equation (7), and approximating $\sum_j k_j(t_k)$ with $2mt_k$, the probability that the node i is attached to node k is given by

$$p_{i,k} = \frac{m}{2}\frac{1}{\sqrt{t_i t_k}}. \quad (8)$$

Moreover, the probability that a node k is connected with nodes i and j is proportional to the probability that i and j are already connected, i.e.

$$p_{i,k}p_{j,k} = m\frac{1}{2t_k}p_{i,j}. \quad (9)$$

By replacing this result in (5) and by the definition (6), we obtain

$$\frac{\partial \langle N_h(t)\rangle}{\partial t} = \frac{m}{2t}(h-1)\langle N_{h-1}(t)\rangle . \quad (10)$$

Consequently, the rate at which new cycles of size h are introduced in the system is proportional to the mean number of cycles of size $h-1$.

Equation (10) has a recursive structure that allows its integration without any detailed information about the probabilities $P^h_{i,j}(t)$. In fact, the rate at which new cycles of length h are formed can be expressed only in terms of the number of cycles of minimal size (i.e. $h=3$),

$$\frac{\partial^{h-3}\langle N_h(\zeta)\rangle}{\partial \zeta^{h-3}} = (h-1)!\,\langle N_3(\zeta)\rangle \quad (11)$$

with $\zeta = \frac{m}{2}\log(t)$ and $h>3$. The number of triangular cycles $\langle N^3(\zeta)\rangle$ can be computed directly, since the triangular cycles are increasing in time following (5),

$$\frac{\partial \langle N_3(t)\rangle}{\partial t} = \frac{1}{2}\sum_{i=1}^{t}\sum_{j=1}^{t} p_{i,k}p_{j,k}p_{i,j}. \quad (12)$$

We compute $\langle N_3(t)\rangle$ using for $p_{i,j}$ the form given by equation (8), approximating sums by integrals we write the rate equation in the form

$$\frac{\partial \langle N_3(t)\rangle}{\partial t} = \frac{1}{2}\left(\frac{m}{2}\right)^3\int_1^t dt_i\int_1^t dt_j\frac{1}{t_i}\frac{1}{t_j}\frac{1}{t} = \frac{1}{2}\left(\frac{m}{2}\right)^3\frac{1}{t}[\log(t)]^2. \quad (13)$$

Integrating (13) we find, in agreement with [22,23],

$$\langle N_3(t)\rangle = \frac{1}{3!}\left[\frac{m}{2}\log(t)\right]^3. \quad (14)$$

Using equation (14) in equation (11), we compute the number of cycles of size h, $\langle N_h(t)\rangle$ and we find

$$\langle N_h(t)\rangle = \left[\frac{m}{2}\log(t)\right]^h(1/h+O(\zeta^{-1})). \sim\left[\frac{m}{2}\log(t)\right]^h \quad (15)$$

with t equal to the total number of nodes N.

As a final comment we observe that in [24] it is derived for the rigorous mathematical version of the BA model (the Linearized Chord Diagram –LCD– model) the same asymptotic scaling of (15) by other means. A more careful analysis concern the result regarding the number of triangles in the LCD model found in [24]. The author of [24] in fact obtain a formula which differ from equation (14) by a factor $(m-1)(m+1)/m^2$. While this correction to equation (14) capture some effect of the correlations between the way the links are connected, not included in the shown derivation, we will see in the next chapter (when this heuristic results will be compared with simulations) that both formula (14) and (15) have to be considered as asymptotic indications of the scaling of the number of triangles and h-cycles present in the network and not as exact predictive calculations of these cycles. In this sense the presented results and the results of [24,25] completely agree.

3 Numerical results for some examples of off-equilibrium scale-free networks

3.1 Direct measurement of N_h for $h = 3, 4, 5$

The expression for the scaling of $N_h(t)$ with the system size t in a BA network does not suggest a practical way to measure $N_h(t)$. To this purpose, one has to study the symmetrical adjacency matrix a of the network, whose generic element a_{ij} is defined by $a_{ij}=1$ if i and j are connected and $a_{ij}=0$ if i and j are not connected. Knowing this matrix, one directly measures the number of paths starting from a node i and returning on it after h steps that visit intermediate nodes only once (in fact the total number of possible paths of size h going from node i to node j are given by the matrix element $(a^h)_{i,j}$ [20]). According to this argument, the term $N_h(t)$ has a dominating term of the type $\sum_i(a^h)_{i,i}/(2h)$ and sub-dominant terms excluding all trivial contributions coming from paths intersecting on themselves. Let us assume that the network does not contains self cycles, i.e. $a_{ii}=0$ for all i of the network. In this case, for $h=3$ we simply have

$$N_3 = \frac{1}{6}\sum_i(a^3)_{ii}. \quad (16)$$

For $h=4$ we have

$$N_4 = \frac{1}{8}\left[\sum_i(a^4)_{ii} - 2\sum_i(a^2)_{ii}(a^2)_{ii} + \sum_i(a^2)_{ii}\right]. \quad (17)$$

To prove equation (17), we observe that in order to find N_4 it is necessary to subtract from $a^4_{i,i}$ all the paths that, going through nodes i_1, i_2, i_3, either have $i_2 = i$ ($\forall i_1, i_3$, either have $i_3 = i_1$ (with the condition $i_2 \neq i$). In Figure 1 we give a graphical representation of the cycles that have to be excluded from the calculation of N_4. From similar considerations it is straightforward for find for N_5,

$$N_5 = \frac{1}{10}\left[\sum_i(a^5)_{ii} - 5\sum_i(a^2)_{ii}(a^3)_{ii} + 5\sum_i(a^3)_{ii}\right]. \quad (18)$$

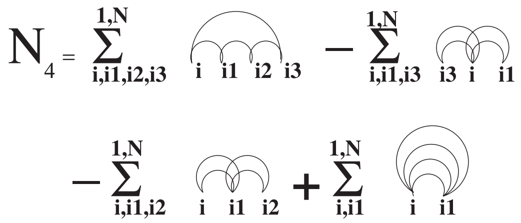

Fig. 1. Calculation of the numbers of cycles of size 4. From all the paths going from i back to i through nodes $i1, i2, i3$ we should remove all the paths that have $i2 = i$ ($\forall i_1, i_3$) and then all the paths that have $i_3 = i1$ (but do not have also $i2 = i$ because they have already been counted by the previous term).

Using relations (16, 17, 18), we can directly measure $N_h(t)$ for $h = 3, 4, 5$ for any growing network.

3.2 The BA model

First off all we measure number of cycles up to size 5 in the BA scale-free network model and we compare our results with the analytic predictions of the previous chapter. The average results obtained from 50 realizations of a BA network of size up to $N = 10^4$ nodes are reported in Figure 2. It is shown there that the scaling of $N_h(N)$ as the power-law of the logarithm of the system size

$$N_h(N) \sim (m \log(N))^{\psi(h)} \tag{19}$$

is directly verifiable. Nevertheless the exponent $\psi(h)$ of the best fit of the numerical curve with the formula (19) differs somewhat from the theoretical expected value equation (15) being $\psi(h)$ slightly less than h, i.e. $\psi(h) < h$ for h up to 5 (see inset of Fig. 2). We believe this is due for the fact that the asymptotic limit in which is valid equation (15) is reached very slowly by the system. For an equilibrium network with $\gamma = 3$ the number of cycles goes like equation (1) with $\xi = 0$. Consequently the scaling (19) can be viewed as a logarithmic correction to the power-law scaling equation (1) with $\xi = 0$.

3.3 The bosonic network

We considered the bosonic network (BN) [26] where each node i is assigned an innate quality, represented by a random 'energy' ϵ_i drawn from the probability distribution $p(\epsilon_i)$. The attractiveness of each node i is then determined jointly by its connectivity k_i and its energy ϵ_i. In particular, the probability that node i acquires a link at time t is given by

$$\Pi_i = \frac{e^{-\beta\epsilon_i} k_i(t)}{\sum_j e^{-\beta\epsilon_j} k_j(t)}, \tag{20}$$

i.e. low energy and high degree nodes are more likely to acquire new links. The parameter $\beta = 1/T$ in Π_i tunes the

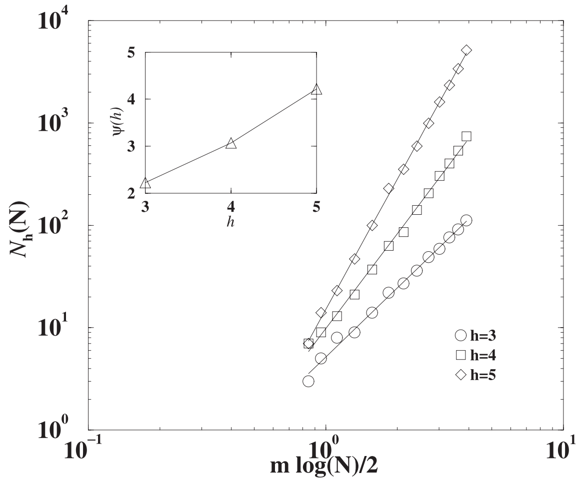

Fig. 2. Scaling of the number of cycles up to size 5 in a BA network as a function of the system size. Data are shown for a network with $m = 2$ and a system size up to $N = 10^4$. In the inset we report the value of the exponent $\psi(h)$ of the best fit following the predicated scaling (22).

relevance of the quality with respect to the degree in the probability of acquisition of new links. Indeed, for $T \to \infty$ the probability Π_i does not depend any more on the energy ϵ_i and the BN model reduces to the Barabási-Albert (BA) model. On the other hand, in the limit $T \to 0$ only the lowest energy node has non zero probability to acquire new links. In reference [26] it has been shown that the connectivity distribution in this network model can be mapped on the occupation number in a Bose gas. According to this analogy, one would expect a corresponding phase transition in the topology of the network at some temperature value T_c. In fact, for energy distributions such that ($p(\epsilon) \to 0$ for $\epsilon \to 0$), one observes a critical temperature T_c. For $T > T_c$ the system is in the "fit-get-rich"(FGR) phase, where nodes with lower energy acquire links at a higher rate that higher energy nodes and the connectivity distribution follows a power-law $P(k) \sim k^{-\gamma}$ with $2 < \gamma \leq 3$, while for $T < T_c$ $\gamma \sim 2$ and a "Bose-Einstein condensate"(BEC) or "winner-takes-all" phase emerges, where a single node grabs a finite fraction of all the links. We simulated this model assuming

$$p(\epsilon) = (\theta + 1)\epsilon^{\theta} \text{ and } \epsilon \in (0, 1) \tag{21}$$

where $\theta = 0.5$. By tuning T there is a changement in the behavior of N_h in the bosonic network from a scaling of the type

$$N_h(N) \sim [\log(N)]^{\psi(h)} \tag{22}$$

demonstrated exact in the $\beta = 0$ limit (for the BA network model) [13], to a scaling of the type

$$N_h(N) \sim N^{\xi(h)} \tag{23}$$

valid a low temperature limit. In reference [13] we claimed that the behavior change right at the Bose-Einstein condensation temperature T_c. A careful analysis of the transition shows indeed that the transition is not so sharp at

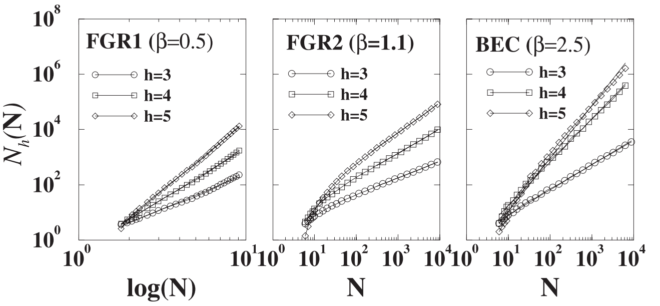

Fig. 3. We report $N_h(N)$ in the three regions of the phase space of the bosonic network with $m = 2$ and $\theta = 0.5$. The chosen points are at $\beta = 0.5$ for the FGR1 phase; at $\beta = 1.1$ for the FGR2 and at $\beta = 2.5$ for the BEC phase. The data shown in figure are averaged over 50 runs.

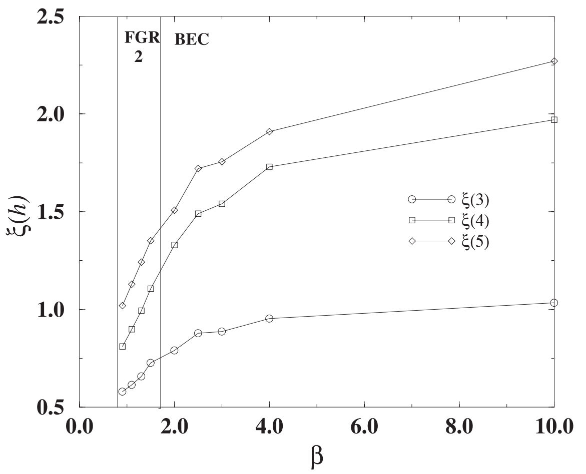

Fig. 4. The exponents $\xi(h)$ that better approximate the behavior of $N_h(N)$ for the bosonic network in the phases FGR2 and BEC are plotted as a function of the inverse temperature β.

T_c but rather smooth. In Figure 3 we show a selection of an extensive study of the scaling of $N_h(N)$ for a network with $m = 2$ and $\theta = 0.5$ $T_c = 0.58(\beta_c = 1.7)$ [26]. The data are collected for network with up to 10^4 nodes and averaged over 50 runs. Analyzing numerically the results we could distinguish three phases: a "high temperature" or "Fit-Get Rich"(FGR1) phase without the condensate and with a scaling of $N_h(N)$ better approximated by (22), a "low temperature" or "Bose-Einstein condensate"(BEC) phase where the network is condensate and $N_h(N)$ scales as (23) and finally an intermediate phase where the network is not yet condensate but for large N the behavior of $N_h(N)$ is better approximated by the power-law equation (23) (FGR2 phase). In Figure 4 we report the values of the exponents $\xi(h)$ that better fit the data in the FGR2 and BEC phases. We observe here that in the phase FGR2 one should expect $\xi(h) \to 0$ at the transition temperature, nevertheless because we are dealing with numerical data in a finite range of system sizes, as we approach the transition we observe that the range in which the power-law fit is statistically reasonable is decreasing. Consequently one finds a power-law exponent $\xi(h)$ that remains significantly different from zero since one actually measure the best power-law fit in smaller and smaller region of large system sizes. This confirms the difficulty in finding the exact transition temperature for the changing behavior of $N_h(N)$ since simulations with very large system sizes would be required. On the contrary in the BEC phase the power-law exponents that we have measured describe the scaling of N_h for at least three order of magnitude. A relevant remark is that in the phases FGR2 and BEC the exponent $\xi(h)$ varies significantly with h and is not bounded by 1. Consequently the scaling of the number of cycles in a bosonic network cannot be explained in terms of the prediction (1) for equilibrium scale-free networks.

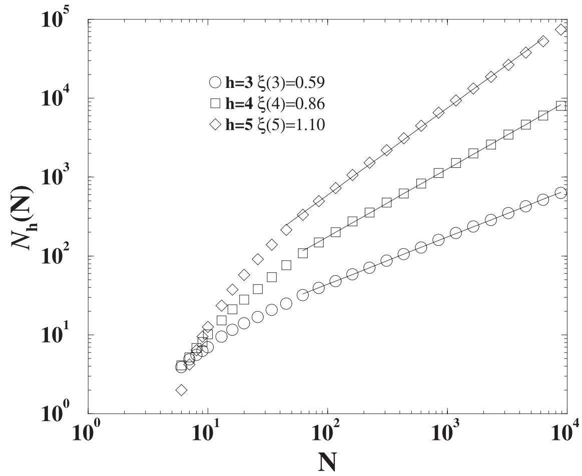

Fig. 5. Scaling of the number of cycles N_h of length h with the system size N. Triangles, rectangles and pentagons asymptotically in N increase like a power-law of the system size with exponents $\xi(h)$ dependent on the length h of the cycles ($\xi(3) = 0.59 \pm 0.02$, $\xi(4) = 0.86 \pm 0.02$, $\xi(5) = 1.10 \pm 0.02$).

3.4 The fitness model

The fitness model [27] has been considered a first approximation of a model for the Internet structure [5,6]. It is a growing network model in which at each time a new node arrives in the network and it is connected by m links to the rest of it. Each node has a fitness η_i extracted randomly from a uniform distribution between zero and one by which it attracts new links. The probability for each node to acquire a new link is given by

$$\Pi_i = \frac{\eta_i k_i(t)}{\sum_j \eta_j k_j(t)}. \quad (24)$$

The resulting network built on a procedure "Good gets richer" is a power-law network with an exponent given by $\gamma = 2.255$. The fitness network has been found also [5,6] to share not trivial correlations and shows a nontrivial scaling of the clustering coefficient $c_3(k)$ and average connectivity of the nearest neighbor $k_{nn}(k)$ with the connectivity k of the considered node.

In Figure 5 we show the scaling of N_h as a function of the system size for the fitness model with $m = 2$. For large N the scaling follows a power-law with exponents $\xi(3) = 0.59 \pm 0.02$, $\xi(4) = 0.86 \pm 0.02$, $\xi(5) = 1.10 \pm 0.02$. Also in this case the equilibrium theory equation (1) fails in describing the cycles structure of this off-equilibrium network.

3.5 Aging nodes network

We have measured the scaling of $N_h(t)$ with the system size t also for a growing network with aging nodes (AN) introduced in [28,29]. The model has been motivated by the observation that in many real networks, e.g. the scientific citations network, old nodes are less cited than recent ones. In the goal of representing this feature, the probability Π_i to attach a link to a node i arrived in the network at time t_i is modified to be

$$\Pi_i = \frac{(t-t_i)^{-\alpha} k_i(t)}{\sum_j (t-t_j)^{-\alpha} k_j(t)} \tag{25}$$

where α is an external parameter. As in the BA model, a new node is connected to m existing nodes. The resulting structure of such a network strongly depends on the constant α. For $\alpha < 1$, the degree distribution is a power law $P(k) \sim k^{-\gamma}$ with an exponent monotonically increasing from $\gamma = 2$ in the limit $\alpha \to -\infty$ to $\gamma \to \infty$ in the limit $\alpha \to 1$: on the other hand, for $\alpha > 1$ no power-law is observed in the degree distribution. Therefore, this model reproduces a scale-free network only in the region $\alpha > 1$.

Moreover, as observed in [28,29], in the limit $\alpha \to -\infty$ the oldest node is connected to an increasing fraction of all the links, reminding the "condensation" observed in the bosonic network. To take into account this phenomenon, we introduce a value α^* such that for $\alpha < \alpha^*$ the fraction of links attached to the most connected node exceeds a finite threshold F. We expect the scaling of $N_h(t)$ to be different in the three regions $\alpha > 1$, $\alpha \in (\alpha^*, 1)$ and $\alpha < \alpha^*$. We measured the total fraction of links $k_{max}/(mN)$ attached to the oldest node in a network with $m = 2$ and $t = 10^4$ nodes. The threshold has been fixed at $F = 0.1$, in order to distinguish the 'condensate' phase from the simple scale-free phase. The value for α^* was found to be $\alpha^* = -1$. We then measured the number of cycles of size $h = 3, 4, 5$ for networks made of up to 10^4 nodes in the three ranges of value of α. We have observed that, for $\alpha > 1$, the number of cycles of size h scales linearly with t (at least for $h = 3, 4, 5$). In the inset (a) of Figure 6 we report the data for $\alpha = 1.5$. On the contrary, for $\alpha \in (-\alpha^*, 1)$ we measured the scaling

$$N_h(N) \sim [\log(N)]^{\psi(h)} \tag{26}$$

with $\psi(h)$ a monotonic function of h. In inset (b) of Figure 6, data for $\alpha = 0.5$ are reported. Finally, in the region $\alpha < \alpha^* = -1$, $N_h(t)$ becomes proportional to a power-law of the system size and the fit is valid for all values of N and not only asymptotically for large N, as it is shown in inset (c) of Figure 6, referring to the case $\alpha = -5$.

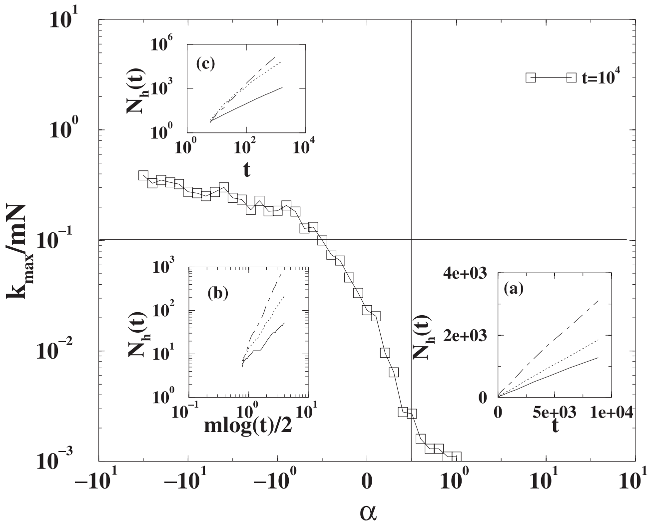

Fig. 6. The order parameter $k_{max}/(mN)$ for a network with aging of the nodes, size $N = 10^4$ and $m = 2$. We distinguish between three region of the phase space:$\alpha > 1$, $\alpha \in (-1, 1)$ and $\alpha < -1$. In the insets we report the typical behavior of $N_h(t)$ as a function of t for $h = 3, 4, 5$ in the three regions.

3.6 Growth and deactivation model

As a final example of off-equilibrium scale-free network we consider the growth and deactivation model [30]. Also this model is motivated by the finding that in the scientific citation network recent papers are more likely acquire new links than older ones. In the network nodes are distinguished between active nodes (that are able to acquire new links) and inactive nodes (that no not acquire new links anymore). The initial condition is a fully connected graph of m active nodes ($k_i = m - 1 \; \forall i = 0, \ldots m-1$). At each time a new node i is added to the network and connected to the m active nodes of the network. Consequently the initial connectivity of each node i is $k_i = m$ and each active node node j increases its connectivity by one, i.e. $k_j \to k_j + 1$. When the node i is linked to the network it is activated while one of the active nodes is deactivated. The probability Π_j that the node j is deactivated is given by

$$\Pi_j = \frac{K-1}{k_j}\;, \tag{27}$$

where the normalization factor is defined as $K - 1 = \left(\sum_{l \in \mathcal{A}} \frac{1}{k_l}\right)^{-1}$. The summation runs over the set $\mathcal{A}$ of the currently active nodes. The network generated in this way is a scale-free network with $\gamma = 3$ [30]. We simulate the network with $m = 10$ and we report the scaling of the number of the cycles with the system size in Figure 7. We observe that the number of cycles $N_h(N)$ grows linearly with N for all h up to length $h = 5$ (the power-law fit to the data give the exponents $\xi(h) = 1.00 \pm 0.05$). Thus for this model the exponents $\xi(h)$ are equal 1 for $h = 3, 4, 5$ and $\xi(h)$ are not dependent on h. Nevertheless the values of $\xi(h)$ are one more time not predicted by the equilibrium theory (1) which gives $\xi(h) = 0$ for $\gamma = 3$.

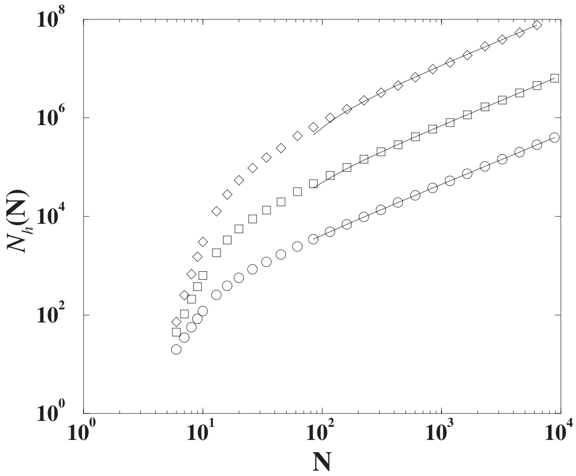

Fig. 7. Scaling of the number of cycles with the network size N in the growth and deactivation model with $m = 10$. The scaling of the cycles up to length $h = 5$ scales linearly with the system size at sufficiently large values of N. We indicate by solid lines the linear fit to the simulation data.

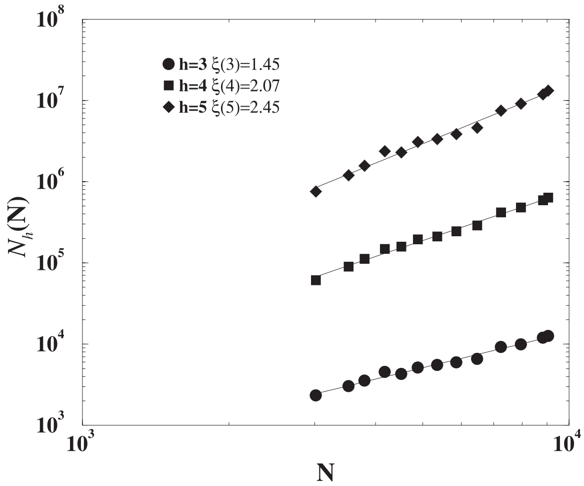

Fig. 8. Scaling of the number of cycles with the network size N in the growth and deactivation model with $m = 10$. The scaling of the cycles up to length $h = 5$ scales linearly with the system size at sufficiently large values of N.We indicate by solid lines the linear fit to the simulation data.

3.7 Comment

The equilibrium networks considered in [14] are all the networks with given connectivity distribution. Thus they include also the special network configurations generated by off-equilibrium scale-free network models. As a random number do not generally coincide with its average value we have found in this paper that the off-equilibrium networks present a scaling of the number of cycles with the system size different from the average scaling of equilibrium networks described in [14] by equation (1). This result is not surprising but it might be interesting because real evolving scale-free networks do not necessarily investigate all the space of equilibrium scale-free networks with their dynamics. This seems to be the case of the evolution of the protein interaction network [31] in which some subgraphs are preserved at high degree during evolution. Another case of a real evolving scale-free network for which data are available is represented by the Internet at the Autonomous System Level which is the object of the next section.

4 The real case of the Internet at the Autonomous System Level

The Internet is an example of a real undirected scale-free network that grows with time and whose historical evolution has been recorded. The data of the Internet at the Autonomous System Level are collected by the University of Oregon Route Views Project and made available by the NLANR (National Laboratory of Applied Network Reasearch). The subset we used in this manuscript are mirrored at COSIN webpage `http://www.cosin.org`. We considered 13 snapshot of the Internet network at the AS level at different times starting from November '97 (with a total number of nodes $N = 3015$) end ending on January '01 (with $N = 9048$). During this time the connectivity distribution follows a power-law with a nearly constant exponent $\gamma \simeq 2.22(1)$ and a almost constant average connectivity $\langle k \rangle \simeq 3.5$. Consequently it is meaningful to measure how the number of cycles up to size 5 present in such network evolve in time. Cycles in the Internet are important because they allow flexibility of the protocols by which the packets are sent to destination. In fact, in the presence of cycles for any given packet the routers can choose in between different paths, trying to minimize congestion. In the absence of cycles the protocols would be fixed and there will be no room for the routers to avoid congestion. In Figure 8 we show the scaling of the number of triangles, quadrilaterals and pentagons N_h with $h = 3, 4, 5$ with the system size N given by the number of nodes present in the Internet at the AS level. We observe a behavior that can be described by the following power-law

$$N_h(N) \sim N^{\xi(h)} \tag{28}$$

with exponents $\xi(3) = 1.45 \pm 0.07$, $\xi(4) = 2.07 \pm 0.01$ and $\xi(5) = 2.45 \pm 0.01$ bigger than 1 and strongly dependent on h, i.e. not constant on h and bound by 1 as the theory equation (1) for equilibrium scale-free networks would suggest.

Consequently we can state that cycles up to size 5 are more frequent in the Internet at the AS level than in a random network with the same connectivity distribution and indeed could play the role of characteristics motifs of the Internet.

In reference [32] we compare this behavior with the behavior of different Internet models, mainly the models formulated in the statistical physicist community (for a complete view of all the Internet models see [19]) the fitness model [27], the Generalized Network Growth model [33] and the bosonic network [26] with same average connectivity. We observe there that while these models capture qualitatively the features of Internet they quantitatively fails in explaining the high value of the exponents $\xi(h)$ and the high density of cycles in the Internet. Further research should investigate if self-organizing dynamical properties can be sufficient to justify the multiplicity of cycles in the Internet or if it is necessary to include some indication of an external design that favors cycles in the evolution of such system.

5 Conclusions

In conclusion we have presented a study of the frequency of cycles in off-equilibrium scale-free networks. We have reported the analytic result obtained for the scaling of the number of cycles with the system size for the BA model which represent the prototype of all off-equilibrium networks. Subsequently we have illustrated an extensive numerical study of the number of cycles in different off-equilibrium network models. We were able to show that as these networks grow, the number of cycles follows a general scaling different from the one predicted for equilibrium scale-free networks with same connectivity distribution. We made the hypothesis that this different behavior of the number of cycles N_h in off-equilibrium scale-free networks can be relevant in order to describe the cycle structure of real networks. In order to prove this hypothesis we have measured the number of cycles in the Internet at the Autonomous System Level and we have observed the anomalous exponents which describe the growth of the number of cycles present in it. Future research will address the two still open problems of the analytical calculation of the number of cycles in off-equilibrium networks and of the formulation of a model that would quantitatively reproduce the cycle structure of the Internet at the Autonomous System level.

The author is grateful to Uri Alon, Guido Caldarelli, Andrea Capocci, Shalev Itzkovitz and Alessandro Vespignani for useful comments and discussions.

References

1. R. Albert, A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002)
2. S. Strogatz, Nature **410**, 268 (2001)
3. S.N. Dorogovtsev, J.F.F. Mendes, *Evolution of Networks* (Oxford University Press, Oxford, 2003)
4. R. Albert, H. Jeong, A.-L. Barabási, Nature **406**, 378 (2000)
5. R. Pastor-Satorras, A. Vázquez, A. Vespignani, Phys. Rev. Lett. **87**, 2587011 (2001)
6. A. Vásquez, R. Pastor-Satorras, A. Vespignani, Phys. Rev. E **65**, 066130 (2002)
7. S. Maslov, K. Sneppen, Science **296**, 910 (2002)
8. S. Maslov, K. Sneppen, A. Zaliznyak, `cond-mat/0205379` (2002)
9. D.J. Watts, S.H. Strogatz, Nature **393**, 440 (1998)
10. G. Caldarelli, R. Pastor-Satorras, A. Vespignani, `cond-mat/0212026` (2002)
11. E. Ravasz, A.L. Somera, D.A. Mongru, Z.N. Oltvai, A.-L. Barabási, Science **297**, 1551 (2002)
12. R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, U. Alon, Science **298**, 824 (2002)
13. G. Bianconi, A. Capocci, Phys. Rev. Lett. **90**, 078701 (2003)
14. S. Itzkovitz, R. Milo, N. Kashtan, G. Ziv, U. Alon, Phys. Rev. E **68**, 026127 (2003)
15. A.-L. Barabási, R. Albert, Science **286**, 509 (1999)
16. A.-L. Barabási, R. Albert, H. Jeong, Physica A **272**, 173 (1999)
17. M. Faloutsos, P. Faloustsos, C. Faloutsos, ACM SIGCOMM '99 Comput. Commun. Rev. **29**, 251 (1999)
18. G. Caldarelli, R. Marchetti, L. Pietronero, Europhys. Lett. **52**, 386 (2000)
19. S.-H. Yook, H. Jeong, A.-L. Barabási, P. Natl. Acad. Sci. USA **99**, 13382 (2002)
20. B. Bollobás, *Random Graphs* (Accademic Press, London, 1985)
21. P. Erdös, A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960)
22. P.M. Gleiss, P.F. Stadler, A. Wagner, D.A. Fell, Adv. Complex Syst. **4**, 207 (2001)
23. K. Klemm, V. Eguíluz, Phys. Rev. E **65**, 057102 (2002)
24. B. Bollobás, O.R. Riordan, in *Handbook of Graphs and Networks* edited by S. Bornholdt, H.G. Schuster (Wiley-VCH, Berlin, 2002), pp. 1–34
25. G. Szabó, M. Alava, J. Kertész, Phys. Rev. E **67**, 056102 (2003)
26. G. Bianconi, A.-L. Barabási, Phys. Rev. Lett. **86**, 5632 (2001)
27. G. Bianconi, A.-L. Barabási, Europhys. Lett. **54**, 436 (2001)
28. S.N. Dorogovtsev, J.F.F. Mendes, Phys. Rev. E **62**, 1842 (2000)
29. S.N. Dorogovtsev, J.F.F. Mendes, Phys. Rev. E **63**, 056125 (2001)
30. K. Klemm, V.M. Eguiluz, Phys. Rev. E **65**, 036123 (2002)
31. S. Wuchty, Z.N. Oltvai, A.-L. Barabási, Nature Genetics **35**, 176 (2003)
32. G. Bianconi, G. Caldarelli, A. Capocci, `cond-mat/0310339` (2003)
33. G. Caldarelli, P. De Los Rios, L. Pietronero, `cond-mat/0307610` (2003)